On the linear discrepancy model and risky shifts in group behavior: a nonlinear Fokker–Planck perspective

2009 J. Phys. A: Math. Theor. 42 155001

(http://iopscience.iop.org/1751-8121/42/15/155001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 155001 (13pp) doi:10.1088/1751-8113/42/15/155001

On the linear discrepancy model and risky shifts in group behavior: a nonlinear Fokker–Planck perspective

T D Frank

Center for the Ecological Study of Perception and Action, Department of Psychology, University of Connecticut, 406 Babbidge Road, Storrs, CT 06269, USA

Received 5 December 2008, in final form 11 February 2009
Published 20 March 2009
Online at stacks.iop.org/JPhysA/42/155001

Abstract
Using a nonlinear Fokker–Planck perspective we re-formulate the linear discrepancy model proposed by Boster and colleagues that describes the emergence of risky shifts during group decision making. Analytical expressions for the stationary case are derived and risky shifts are obtained by Monte Carlo simulations. Striking similarities with the Kuramoto model for group synchronization are pointed out.

PACS numbers: 05.40.−a, 02.50.Ga, 89.65.−s

1. Introduction

Several efforts have been made to understand human group behavior from a physicist's point of view [1–6]. The reason for this is that group behavior in general and group decision making in particular emerges due to interactions between group members. Consequently, taking a physicist's perspective, in such instances we are dealing with many-body systems composed of interacting subsystems. For example, interaction in terms of communication and in terms of confrontation of individuals with group standards determines decision making in groups and the emergence of social conformity [7–10]. Synchronization during clapping is established by acoustic couplings between group members [4]. Bridge vibrations can be excited by walkers who walk in synchronization and are coupled with each other indirectly via the very same vibrational modes that they excite [5]. In particular, similarly to mean field theory the dynamics of group behavior can be captured by means of nonlinear Markov processes [11–20] as described by nonlinear Fokker–Planck equations [21–33].

In a previous study, the nonlinear Markov perspective has been applied to propose a theoretical model for the emergence of social conformity [6]. The aim of the present study is to address another important phenomenon in social psychology from the perspective of nonlinear Markov processes: the emergence of risky shifts (or group polarization) during group decision making. Let us assume a group discusses an issue and exhibits at the beginning of the discussion on the average a mildly risky attitude. In several experiments it has been shown that after the discussion the average attitude can change in the direction of a more risky attitude [10]. This is the so-called risky shift. It should be mentioned that also the opposite phenomenon can be observed. A group that initially exhibits a moderately cautious attitude becomes even more cautious during the discussion. Consequently, in the literature the risky shift and the cautious shift are regarded as two instances of a polarity shift that arises from interactions between group members [10]. Boster and colleagues [34–36] proposed a linear discrepancy model to describe the emergence of risky shifts. The objective of our study is to re-formulate this model in terms of a nonlinear Markov process defined by a nonlinear Fokker–Planck equation. In doing so, we will supplement the model with a particular fluctuating force (which has been neglected in [34]), we will obtain an analytical expression for the stationary statistical properties of the group decision process, and we will be able to identify analogies to a benchmark model for group synchronization: the Kuramoto model [25].

2. Nonlinear Fokker–Planck modeling of the emergence of risky shifts

2.1. Linear discrepancy model

Along the same lines as the suggestion by Boster and colleagues [34], we will assign real numbers to arguments and opinions. That is, we will put opinions on a continuous scale or interval Ω. Let X_i denote the opinion of a member i participating in a group discussion involving N participants. By convention we say that $X_i = 0$ corresponds to a neutral opinion, whereas $X_i > 0$ ($X_i < 0$) describes a risky (cautious) opinion. Moreover, we assume that larger positive X_i values describe riskier opinions than smaller positive X_i values. For the sake of simplicity, we will define opinions X_i on the whole real line: $\Omega = \mathbb{R}$. Let t denote time measured on a continuous scale. The beginning of a group discussion session will be denoted by $t = 0$. Accordingly, the trajectory $X_i(t)$ describes the opinion of a participant during group decision making. In order to apply concepts of mean field theory to group decision making, we neglect finite-size effects and will consider large groups in the limiting case $N \to \infty$. In this case, the function $\rho(x, t) = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \delta(x - X_i(t))$ (where $\delta(\cdot)$ is the Dirac delta function) corresponds to a probability density and describes how opinions are distributed in a discussion group under consideration. Likewise, $\langle X(t)\rangle_\rho = \int_\Omega x\rho(x, t)\,\mathrm{d}x$ denotes the mean opinion of the group at time t. We will refer to $\langle X(t)\rangle_\rho$ as group opinion. As suggested in [34], we assume that participants in general do not present their arguments and opinions with the same frequency. We assume that a speaker that has an opinion consistent with or close to the group opinion presents his or her argument more frequently than a speaker who holds an opinion that deviates to a large extent from the group opinion. Let $f(x, \langle X\rangle_\rho)$ denote the frequency with which a speaker (a member of the discussion group) expresses his or her opinion X given that the group exhibits a group opinion $\langle X\rangle_\rho$. In analogy to the model proposed in [34], we assume that the opinion $X(t)$ of a representative group member satisfies the stochastic evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\alpha \int_\Omega f(x', \langle X(t)\rangle_\rho)(x - x')\rho(x', t)\,\mathrm{d}x' + \sqrt{Q}\Gamma(t). \tag{1}$$

According to equation (1), a discussant with opinion X changes his or her opinion when confronted with the opinion X' of another discussant. The change is proportional to the linear discrepancy between the two opinions measured in terms of $X - X'$ and tends to reduce the discrepancy. The parameter $\alpha > 0$ is the proportional factor relating the opinion change $\mathrm{d}X/\mathrm{d}t$ to the discrepancy $X - X'$. The linear discrepancy $X - X'$ is weighted with the percentage $\mathrm{d}\rho = \rho(x', \cdot)\,\mathrm{d}x'$ of group members that exhibit opinion X'. In addition, the discrepancy

$X - X'$ is weighted with the frequency f that members of the discussion group with opinion X' indeed actively advance their opinions (and not just passively listen to the arguments made by others). The total deterministic impact of all group members on the representative group member is given by the integral $-\alpha \int_\Omega f(x', \langle X(t)\rangle_\rho)(x - x')\rho(x', t)\,\mathrm{d}x'$. The product $\sqrt{Q}\Gamma(t)$ in the stochastic evolution equation (1) describes a fluctuating force that is composed of the so-called noise amplitude $Q \geqslant 0$ and a time-dependent function $\Gamma(t)$. Note that the term $\sqrt{Q}\Gamma(t)$ corresponds to a so-called additive fluctuating force. We will briefly consider alternative (multiplicative) types of fluctuating forces in the conclusions.

We are interested in modeling group decision making by the most simple but not purely random process. This is a Markov process (see [37] for classification into purely random processes, Markov processes and non-Markov processes). Consequently, we assume that Γ is defined by a Langevin force [37]. As a normalization condition for Γ we use $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$. In order to obtain a closed description we use the mean field approach and replace the ensemble of group members with the statistical ensemble of the representative member X. To this end, we introduce the probability density $P(x, t) = \langle\delta(x - X(t))\rangle$ (note that here and in what follows $\langle\cdot\rangle$ without index ρ denotes averaging with respect to a statistical ensemble). Having defined P, we replace in equation (1) the group opinion distribution ρ by the probability density P of the representative group member X. That is, we put $\rho(x, t) = P(x, t)$. Likewise, we replace $\langle X\rangle_\rho$ by $\langle X\rangle$ (i.e. by $\int_\Omega x P(x, t)\,\mathrm{d}x$). For details see [25, 33, 38]. Then, equation (1) becomes a closed stochastic evolution equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\alpha \int_\Omega f(x', \langle X(t)\rangle)(x - x')P(x', t)\,\mathrm{d}x' + \sqrt{Q}\Gamma(t). \tag{2}$$

Equation (2) is a self-consistent Langevin equation [33]. The corresponding evolution equation for P is given by a nonlinear Fokker–Planck equation [33] and reads

$$\frac{\partial}{\partial t}P(x, t) = \frac{\partial}{\partial x}\alpha \left(\int_\Omega f(x', \langle X(t)\rangle)(x - x')P(x', t)\,\mathrm{d}x'\right) P(x, t) + Q\frac{\partial^2}{\partial x^2}P(x, t). \tag{3}$$

Aiming at an analytical discussion, we model f by a Gaussian function,

$$f(x, \langle X\rangle) = \sqrt{\frac{\lambda}{2\pi}} \exp\left\{-\frac{\lambda}{2}(x - \langle X\rangle)^2\right\}, \tag{4}$$

with $\lambda > 0$ as opposed to the function originally proposed by Boster and colleagues [34]. Note that just as in [34] the function (4) decays monotonically with increasing deviation $|x - \langle X\rangle|$. In sum, equations (3) and (4) define in terms of a nonlinear Fokker–Planck equation a linear discrepancy model that is consistent with the fundamental ideas centered around the originally proposed linear discrepancy model by Boster and colleagues [34].

Before examining the properties of the model given by equations (3) and (4), we note that the self-consistent Langevin equation (2) can alternatively be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\alpha\,(\langle f\rangle X - \langle Xf\rangle) + \sqrt{Q}\Gamma(t), \tag{5}$$

with the expectation values defined by

$$\langle f\rangle = \int_\Omega f(x, \langle X(t)\rangle)P(x, t)\,\mathrm{d}x, \tag{6}$$

$$\langle Xf\rangle = \int_\Omega x f(x, \langle X(t)\rangle)P(x, t)\,\mathrm{d}x. \tag{7}$$

In general, these expectation values $\langle f\rangle$ and $\langle Xf\rangle$ depend on time t. In the stationary case, however, they become constants.

2.2. Stationary case

Our next objective is to determine the stationary probability density $P_{st}(x)$ of the linear discrepancy model defined by equations (3) and (4). In the stationary case, we have the stationary expectation values

$$\langle X\rangle_{st} = \int_\Omega x P_{st}(x)\,dx, \tag{8}$$

$$\langle f\rangle_{st} = \int_\Omega f(x, \langle X\rangle_{st}) P_{st}(x)\,dx, \tag{9}$$

$$\langle Xf\rangle_{st} = \int_\Omega x f(x, \langle X\rangle_{st}) P_{st}(x)\,dx. \tag{10}$$

Likewise, in the stationary case the nonlinear Fokker–Planck equation (3) becomes

$$-\alpha(\langle f\rangle_{st} x - \langle xf\rangle_{st}) P_{st} = Q\frac{d}{dx} P_{st}. \tag{11}$$

Consequently, the stationary probability density is given by

$$P_{st}(x) = \frac{1}{Z}\exp\left\{-\frac{\alpha\langle f\rangle_{st}}{2Q}\left(x - \frac{\langle Xf\rangle_{st}}{\langle f\rangle_{st}}\right)^2\right\}, \tag{12}$$

where Z is a normalization constant such that $\int_\Omega P_{st}(x)\,dx = 1$ holds. The unknown expectation values $\langle X\rangle_{st}, \langle f\rangle_{st}, \langle Xf\rangle_{st}$ can be determined by solving the self-consistency equations

$$\langle X\rangle_{st} = \frac{1}{Z}\int_\Omega x \exp\left\{-\frac{\alpha\langle f\rangle_{st}}{2Q}\left(x - \frac{\langle Xf\rangle_{st}}{\langle f\rangle_{st}}\right)^2\right\} dx, \tag{13}$$

$$\langle f\rangle_{st} = \frac{1}{Z}\int_\Omega f(x, \langle X\rangle_{st}) \exp\left\{-\frac{\alpha\langle f\rangle_{st}}{2Q}\left(x - \frac{\langle Xf\rangle_{st}}{\langle f\rangle_{st}}\right)^2\right\} dx, \tag{14}$$

$$\langle Xf\rangle_{st} = \frac{1}{Z}\int_\Omega x f(x, \langle X\rangle_{st}) \exp\left\{-\frac{\alpha\langle f\rangle_{st}}{2Q}\left(x - \frac{\langle Xf\rangle_{st}}{\langle f\rangle_{st}}\right)^2\right\} dx. \tag{15}$$

(For the concept of self-consistency equations see, e.g., [25, 33].) From equation (13) it immediately follows that $\langle X\rangle_{st} = \langle Xf\rangle_{st}/\langle f\rangle_{st}$. That is, we have a factorization like

$$\langle Xf\rangle_{st} = \langle X\rangle_{st}\langle f\rangle_{st}. \tag{16}$$

Equation (16), in turn, implies that the stationary probability density (12) can be written as

$$P_{st}(x) = \frac{1}{Z}\exp\left\{-\frac{\alpha\langle f\rangle_{st}}{2Q}(x - \langle X\rangle_{st})^2\right\}. \tag{17}$$

Turning to the second self-consistency equation, from equation (14) a detailed calculation yields the intermediate result:

$$\langle f\rangle_{st} = \frac{\alpha\lambda}{2\pi(\lambda Q + \alpha\langle f\rangle_{st})}. \tag{18}$$

In the limiting case $\lambda \to \infty$, we have $f(x, \langle X\rangle) = \delta(x - \langle X\rangle)$ and from equation (18) it follows that $\langle f\rangle_{st} = \alpha/(2\pi Q)$. This implies that

$$P_{st}(x) = \frac{1}{Z}\exp\left\{-\frac{\alpha^2}{4\pi Q^2}(x - \langle X\rangle_{st})^2\right\}, \tag{19}$$

such that the spread of the opinions can be measured in terms of the variance $\sigma^2 = 2\pi Q^2/\alpha^2$. In the general case, we have $\lambda < \infty$. From equation (18) it follows that

$$\langle f\rangle_{\rm st} = -\frac{\lambda Q}{2\alpha} + \sqrt{\frac{\lambda}{2\pi} + \left(\frac{\lambda Q}{2\alpha}\right)^2}. \tag{20}$$

Using this result in combination with the third self-consistency equation (15), we obtain

$$\langle Xf\rangle_{\rm st} = \langle X\rangle_{\rm st}\langle f\rangle_{\rm st}. \tag{21}$$

That is, the third self-consistency equation is redundant and yields the same result as the first self-consistency equation. This indicates that the expectation values $\langle X\rangle_{\rm st}$, $\langle f\rangle_{\rm st}$, $\langle Xf\rangle_{\rm st}$ are not completely defined in terms of the model parameters λ, α, Q. In fact, the linear discrepancy model given by equations (3) and (4) is invariant against translations. If we replace x by $x+\xi$ for an arbitrary real number ξ we obtain equations (3) and (4) again. Consequently, the mean value $\langle X\rangle_{\rm st}$ is not fixed by the model parameters λ, α, Q (we will return to this issue in the conclusions). Rather, the mean value $\langle X\rangle_{\rm st}$ depends on the initial probability density $P(x, 0)$. Let $\rho(x, 0) = u(x)$ denote the initial distribution of opinions in the discussion group under consideration. In line with our mean field approach, we put $\rho(x, 0) = P(x, 0) = u(x)$. Then, we may say that the final stationary group opinion $\langle X\rangle_{\rm st}$ depends on the initial distribution of opinions u, whereas the expectation value $\langle f\rangle_{\rm st}$ does not depend on u and is completely defined by the parameter set $\{\lambda, \alpha, Q\}$.

This situation is in analogy with the Kuramoto model of an infinitely large ensemble of interacting subsystems that have angular state variables defined on the interval $[0, 2\pi]$ [25, 26, 33]. The nonlinear Fokker–Planck equation of the Kuramoto model reads

$$\frac{\partial}{\partial t}P(x,t) = \alpha\frac{\partial}{\partial x}\left(\int_\Omega \sin(x-x')P(x',t)\,{\rm d}x'\right)P(x,t) + Q\frac{\partial^2}{\partial x^2}P(x,t), \tag{22}$$

with $\alpha, Q \geqslant 0$. In the context of the Kuramoto model, the so-called cluster phase and cluster amplitude [33] can be considered as counterparts to the mean and variance of the linear discrepancy model. It is well known that for the Kuramoto model the stationary cluster phase is not entirely fixed by the parameters of the Kuramoto model but depends on the initial distribution of angular variables. In contrast, the stationary cluster amplitude of the Kuramoto model is completely defined in terms of model parameters. For details see [25, 26, 33].

Let us return to the derivation of the stationary probability density of the linear discrepancy model (3). From equation (17) it follows that the opinion variance in the stationary case is given by

$$\sigma^2 = \frac{Q}{\alpha\langle f\rangle_{\rm st}}. \tag{23}$$

Let us first discuss the special case $\lambda \to 0$. In this case, we have $\langle f\rangle_{\rm st} \to 0$ (see equation (20)) which implies that $\sigma^2 \to \infty$. Next, let us consider the general case $\lambda > 0$. To this end, we substitute equation (20) into equation (23). Thus, we obtain

$$\sigma^2 = \frac{2}{\lambda}\frac{1}{\sqrt{1+\frac{2\alpha^2}{\pi\lambda Q^2}} - 1}. \tag{24}$$

In sum, the stationary probability density (19) can be written as

$$P_{\rm st}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\langle X\rangle_{\rm st})^2}{2\sigma^2}\right\}, \tag{25}$$

with σ^2 defined in equation (24).

Table 1. Opinion variance σ^2 for several limiting cases.

$J = \lambda^{-1}, D$	λ^{-1}	D
$J \to 0$	$\sigma^2 \to 2\pi D^2$	$\sigma^2 \to 0$
$J \to \infty$	$\sigma^2 \to \infty$	$\sigma^2 \to \infty$

Let us consider some further limiting cases. We see that for $Q/\alpha \to 0$ we have $\sigma^2 \to 0$. For $Q/\alpha \to \infty$ we have $\sigma^2 \to \infty$. At that stage we may focus on the interpretation of the parameters α and Q. To this end, we first note that we may rescale time t by $t = \alpha\tilde{t}$ and introduce the probability density $\tilde{P}(x,\tilde{t}) = P(x, t = \alpha\tilde{t})$ with respect to the rescaled time frame $\tilde{t}$. Dividing both sides of the nonlinear Fokker–Planck equation (3) by α, we obtain

$$\frac{\partial}{\partial t}\tilde{P}(x,\tilde{t}) = \frac{\partial}{\partial x}\left(\int_\Omega f(x', \langle X\rangle)(x-x')\tilde{P}(x',\tilde{t})\,\mathrm{d}x'\right)\tilde{P}(x,\tilde{t}) + D\frac{\partial^2}{\partial x^2}\tilde{P}(x,\tilde{t}), \tag{26}$$

with $D = Q/\alpha$. The relation $t = \alpha\tilde{t}$ indicates that α defines a characteristic time scale. That is, α may be regarded as a relaxation constant. In contrast, the ratio Q/α which we have denoted above by D describes the diffusive properties of the decision process. With these notions at hand, we can summarize our result as follows. The stationary probability density of the linear discrepancy model (3), (4) is given by

$$P_{\mathrm{st}}(x|u) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\langle X\rangle_{\mathrm{st}})^2}{2\sigma^2}\right\}, \tag{27}$$

with

$$\sigma^2 = \frac{2}{\lambda}\frac{1}{\sqrt{1+\frac{2}{\pi\lambda D^2}}-1}. \tag{28}$$

As indicated by the notation $P_{\mathrm{st}}(x|u)$, the stationary probability density depends on the initial distribution u. More precisely, it is the group opinion $\langle X\rangle_{\mathrm{st}}$ that depends on the initial probability density $u(x)$. In contrast, the opinion variance σ^2 is independent of the initial distribution u of opinions and is completely determined by the parameter λ and the diffusion parameter $D = Q/\alpha$. Furthermore, we have the special cases $\lambda \to 0 \Rightarrow \sigma^2 \to \infty$, $\lambda \to \infty \Rightarrow \sigma^2 \to 2\pi D^2$, $D \to 0 \Rightarrow \sigma^2 \to 0$, $D \to \infty \Rightarrow \sigma^2 \to \infty$.

In order to illustrate how σ^2 depends on λ and D, we use $1/\lambda$ instead of λ. Then σ^2 increases monotonically when $1/\lambda$ and D increase as shown in figure 1. Moreover limiting cases can be summarized as shown in table 1.

2.3. Stability analysis by means of an H-theorem

Stationary solutions of nonlinear Fokker–Planck equations may become unstable at critical parameters. Bifurcations to multi-stable states [25, 33], oscillatory behavior [39, 40] and even chaos [41–43] have been observed. Since stationary solutions of the linear discrepancy model (3) correspond to Gaussian distributions, we will examine the stability of these Gaussian solutions in a two-step approach. First, we will examine the evolution of the mean m and the variance K of discussion groups with initially Gaussian distributed opinions. Second, we will study the stability of the Gaussian stationary distributions of the form (27), (28) in the general case for arbitrary initial distributions.

Let

$$P_G(x,t) = \sqrt{\frac{1}{2\pi K(t)}}\exp\left\{-\frac{[x-m(t)]^2}{2K(t)}\right\} \tag{29}$$

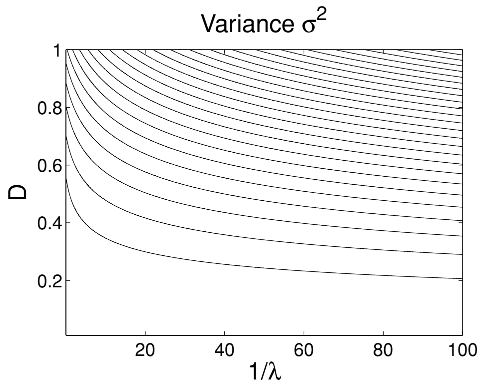

Figure 1. Contour plot of opinion variance σ^2 as defined by equation (28). σ^2 increases monotonically with $1/\lambda$ and D.

denote a Gaussian time-dependent distribution with mean $m(t)$ and variance $K(t)$. From the Langevin equation (2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t) = -\alpha(\langle f\rangle m(t) - \langle Xf\rangle). \tag{30}$$

In section 2.2, we have shown that for Gaussian distributions the equivalence $\langle f\rangle m = \langle Xf\rangle$ holds; see equation (16). Therefore, we have

$$m(t) = m(0), \tag{31}$$

for any $t \geqslant 0$. The mean value is neutrally stable in the sense that if we perturb the mean value by an amount ϵ like $m \to m + \epsilon$ then the mean value will maintain this perturbed value for all times. Having discussed the evolution of the mean value, we turn next to the variance K. The translational invariance of the linear discrepancy model implies that the evolution of the variance K does not depend on the mean value m. That is, for $m \neq 0$ as well as for $m = 0$ we obtain the same functions $K(t)$. For the sake of convenience, we put $m = 0$ in what follows. The evolution equations for $K(t) = \langle X^2(t)\rangle$ can then be computed from the Fokker–Planck equation (3). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = -2\alpha\left(\langle f\rangle K - \frac{Q}{\alpha}\right). \tag{32}$$

Note that in the stationary case we have $\langle f\rangle_{\mathrm{st}}K_{\mathrm{st}} = Q/\alpha$ which is the result obtained earlier (see equation (23)). Using equations (4) and (29), we see that

$$\langle f\rangle = \sqrt{\frac{1}{2\pi[K(t) + 1/\lambda]}}. \tag{33}$$

Consequently, equation (32) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = -\frac{2\alpha}{\sqrt{2\pi}}\underbrace{\sqrt{\frac{K^2}{K + 1/\lambda}}}_{\xi(K)} + 2Q. \tag{34}$$

The function $\xi(K)$ indicated in equation (34) is a strictly monotonically increasing function satisfying $\xi(K) = \sqrt{K}$ for $K \to \infty$ and $\xi(0) = 0$. Moreover, the right-hand side of equation (34) vanishes at $K = \sigma^2$ defined by equation (28) (see section 2.2). Since $\xi(K)$ is strictly monotonically increasing, we have $\mathrm{d}K(t)/\mathrm{d}t > 0$ for $K < \sigma^2$ and $\mathrm{d}K(t)/\mathrm{d}t < 0$ for $K > \sigma^2$. Consequently, the fixed point $K = \sigma^2$ is globally stable, and any Gaussian distribution converges to a stationary solution $P_{\mathrm{st}}(x|u)$ as defined by equations (27), (28):

$$\lim_{t\to\infty} P_G(x,t) = P_{\mathrm{st}}(x|u). \tag{35}$$

The explicit graph $K(t)$ can be obtained by solving equation (34) numerically for any given initial condition $K(0)$. On the one hand, our analysis shows how to compute time-dependent Gaussian solutions $P_G(x,t)$ of the linear discrepancy model. On the other hand, our analysis reveals that the stationary Gaussian probability density given by equations (27), (28) corresponds to a fixed point in the two-dimensional plane spanned by the parameters m and K and is neutrally stable in the m direction and globally stable in the K direction.

Next we construct an H-theorem to show that for arbitrary initial conditions solutions of the linear discrepancy model (3) converge to the time-dependent Gaussian solutions P_G identified above. To this end, we follow previous studies [44, appendix B] and [41]. Let $L(P)$ denote the nonlinear Fokker–Planck operator,

$$L(P) = \alpha \frac{\partial}{\partial x} \int_\Omega f\left(x', \int_\Omega x'' P(x'',t)\,\mathrm{d}x''\right)(x-x')P(x',t)\,\mathrm{d}x' + Q\frac{\partial^2}{\partial x^2}, \tag{36}$$

involving an arbitrary solution $P(x,t)$ of equation (3). Let $w(x,t)$ denote a probability density that solves the linear Fokker–Planck equation

$$\frac{\partial}{\partial t} w(x,t) = L(P) w(x,t). \tag{37}$$

This Fokker–Planck equation describes a time-inhomogeneous process because the solution $P(x,t)$ of the nonlinear Fokker–Planck equation (3) acts as a driving force for the probability density $w(x,t)$. In other words, let $X_w(t)$ denote the random variable of the process with probability density $w(x,t)$. Then, $X_w(t)$ satisfies the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_w(t) = -\alpha\left(\langle f\rangle_P X_w(t) - \langle Xf\rangle_P\right) + \sqrt{Q}\,\Gamma(t), \tag{38}$$

with time-dependent expectation values

$$\begin{aligned} \langle f\rangle_P &= \int_\Omega f(x, \langle X\rangle_P) P(x,t)\,\mathrm{d}x, \\ \langle Xf\rangle_P &= \int_\Omega x f(x, \langle X\rangle_P) P(x,t)\,\mathrm{d}x, \\ \langle X\rangle_P &= \int_\Omega x P(x,t)\,\mathrm{d}x. \end{aligned} \tag{39}$$

We define the functional [37]

$$H(t) = \int_\Omega P(x,t) \ln\left(\frac{P(x,t)}{w(x,t)}\right)\mathrm{d}x \geqslant 0. \tag{40}$$

Since the probability densities w and P satisfy the same Fokker–Planck operator, computation of the time derivative of H yields the following well-known result from the theory of linear Fokker–Planck equations [37]:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = -Q \int_\Omega P(x,t)\left[\frac{\partial}{\partial x} \ln \frac{P(x,t)}{w(x,t)}\right]^2 \mathrm{d}x \leqslant 0. \tag{41}$$

From equations (40) and (41) we conclude that $H(t)$ becomes constant for $t \to \infty$ which implies that the two solutions $P(x,t)$ and $w(x,t)$ approach each other:

$$\lim_{t\to\infty} [P(x,t) - w(x,t)] = 0. \tag{42}$$

We consider next the special case in which $w(x, t)$ at time $t = 0$ corresponds to a Gaussian distribution with mean $m_w(0)$ and variance $K_w(0)$. A detailed calculation shows that the time-inhomogeneous linear Fokker–Planck equation (37) exhibits time-dependent Gaussian distributions

$$w(x,t) = \sqrt{\frac{1}{2\pi K_w(t)}} \exp\left\{-\frac{[x - m_w(t)]^2}{2K_w(t)}\right\}, \tag{43}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} m_w(t) = -\alpha\left(\langle f\rangle_P m_w(t) - \langle Xf\rangle_P\right) \tag{44}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} K_w(t) = -2\alpha\left(\langle f\rangle_P K_w(t) - \frac{Q}{\alpha}\right). \tag{45}$$

As shown in equation (42), in the long-time limit the probability densities P and w approach each other. Consequently, the expectation values $\langle f\rangle_P$, $\langle Xf\rangle_P$ and $\langle X\rangle_P$ approach the expectation values $\langle f\rangle_w$, $\langle Xf\rangle_w$ and m_w, respectively. This, in turn, implies that the structure of the evolution equations (44) and (45) approaches the structure of the evolution equations (30) and (32), respectively. In other words, the Gaussian probability density $w(x, t)$ approaches a Gaussian distribution $P_G(x, t)$ with constant mean and variance K defined by equation (34). Recall that there are infinitely many Gaussian distributions $P_G(x, t)$ that differ with respect to $m(0)$ and $K(0)$. Therefore, the statement above actually means that $w(x, t)$ converges to one member of the infinitely large family of Gaussian distributions $P_G(x, t)$. In this sense, the limiting case

$$\lim_{t\to\infty} [w(x,t) - P_G(x,t)] = 0 \tag{46}$$

holds. Taking equations (35), (42) and (46) together, we see that for arbitrary initial distributions $P(x, 0)$ the solutions $P(x, t)$ of the linear discrepancy model (3) converge to a stationary probability density $P_{\mathrm{st}}(x|u)$ of the form (27), (28):

$$\lim_{t\to\infty} P(x,t) = \lim_{t\to\infty} w(x,t) = \lim_{t\to\infty} P_G(x,t) = P_{\mathrm{st}}(x|u). \tag{47}$$

2.4. Transient behavior and risky shifts

As pointed out by Boster and colleagues [34], the linear discrepancy model predicts that risky shifts emerge in discussion groups whose initial opinions are asymmetrically distributed. For symmetric initial conditions the group opinion is constant. It is clear that the nonlinear Fokker–Planck version (3) and (4) of the linear discrepancy model exhibits these features as well. For initial distributions $u(x + z) = u(z - x)$ symmetric with respect to a particular mean value z, we have $\langle Xf\rangle = 0$ at any time $t \geqslant 0$. Consequently, the mean value $\langle X\rangle$ does not change with time (hint: see the Langevin equation (5); see also section 2.3 on Gaussian distributions). However, for asymmetric initial distributions u in general the expectation value $\langle Xf\rangle$ will differ from zero. Consequently, the mean value $\langle X\rangle$ will evolve as a function of time; see equation (5). In particular, if $\langle X\rangle > 0$ and $\langle Xf\rangle > 0$ then the mean value $\langle X\rangle$ will increase and we obtain a risky shift. In contrast, if $\langle X\rangle < 0$ and $\langle Xf\rangle < 0$ then the mean value $\langle X\rangle$ will become even more negative and we obtain a cautious shift. Figures 2 and 3 illustrate this situation by a numerical example for the risky shift.

We generated an ensemble of random variables that were distributed according to the relatively flat and asymmetric probability density shown in figure 2. This distribution

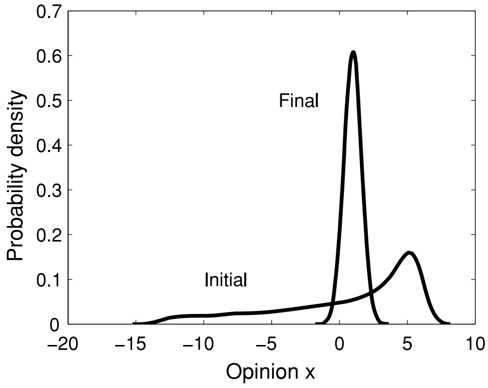

Figure 2. Initial and stationary probability density used in a simulation of the linear discrepancy model (3) and (4). Parameters: $\lambda = 0.5$, $\alpha = 0.5\ \mathrm{min}^{-1}$, $Q = 0.05\ \mathrm{min}^{-1}$.

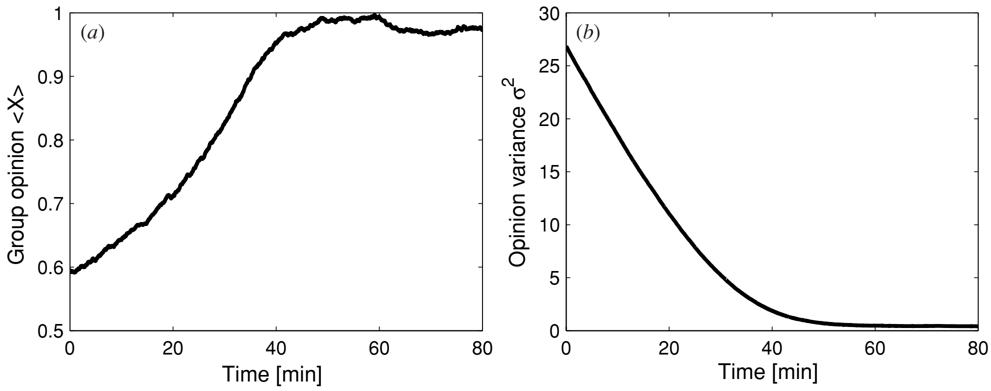

Figure 3. Evaluation of the group opinion $\langle X(t)\rangle$ (left panel) and the opinion variance $\sigma^2(t)$ (right panel) computed from simulated data. Parameters as in figure 2.

corresponds to the initial probability density of group opinions. We then simulated the linear discrepancy model (3) and (4) by solving the Langevin equation (5). To this end, we used the third (self-consistent plus averaging) method described in [33, page 40]. The final and stationary distribution of opinions is shown in figure 2 as a clearly Gaussian distribution. The group opinion (mean value $\langle X\rangle$) as a function of time increased for the parameters used in the simulation and is shown in figure 3 (left panel). That is, figure 3 illustrates the emergence of a risky shift during a decision-making process predicted by the linear discrepancy model. In our simulation, the variance approached a stationary value as well (see the right panel of figure 3). We checked the stationary level of the opinion variance and found that it was in good approximation of the value σ^2 predicted by equation (24).

3. Conclusions

Using a nonlinear Fokker–Planck perspective we re-formulated a model proposed by Boster and colleagues that describes the emergence of risky shifts during group decision making.

We showed that the model predicts that finally the opinions of the group members will be Gaussian distributed. It is clear that the Gaussian shape is a consequence of the assumed linear interaction between group members and the assumption that variability is imposed on the group members in the form of an additive noise term (additive fluctuating force). However, it is well known that the Fokker–Planck approach to stochastic processes also allows us to model nonlinear evolution equations. For example, if the linear discrepancy is replaced by a cubic discrepancy like $-\alpha(x - \langle X\rangle)^3$, then the cubic model would predict opinions distributed like a Boltzmann distribution $P \propto \exp\{-cV\}$ involving a fourth-order potential $V = \alpha(x - \langle X\rangle)^4/4$, where c can be determined by means of Monte Carlo simulation methods.

A multiplicative noise term interpreted according to the Ito calculus [37] would affect the shape of the stationary distribution. For example, replacing $\sqrt{Q}\Gamma$ in the Langevin equation (2) by $\sqrt{Q}X(t)\Gamma$ would yield a model that exhibits stationary power-law distributions. However, the mechanism proposed by Boster and colleagues that results in the risky shift would not be affected by the choice of a multiplicative noise term—as long as the noise source is interpreted according to Ito's perspective and the state-dependent diffusion coefficient of the noise source exhibits reflectional symmetry ($x \leftrightarrow -x$). That is, a multiplicative noise source would not result in a risky shift as such, i.e. a symmetric initial distribution would not show a risky shift irrespective of our choice of a noise term. The risky shift can only emerge if group members have initially opinions that are asymmetrically distributed. This is the mechanism suggested by Boster *et al*. In contrast, a multiplicative noise source interpreted according to the Stratonovich calculus can result in a noise-induced drift and consequently lead to a noise-induced risky shift. Consequently, a challenge for future experimental studies is to test the hypothesis of asymmetry-induced risky shifts against the hypothesis of noise-induced risky shifts.

We revealed a striking similarity between the linear discrepancy model and the Kuramoto model. Both models predict that the variability is independent of the initial distribution. In contrast, the mean opinion or behavior crucially depends on the initial distribution. In other words, we are dealing with multistability. Multistability in turn is a characteristic feature of many-body systems composed of interacting subsystems. In our context, the observation of multistable solutions indicates that group decision making arises due to interaction between group members and does not result from the impact of an external driving force.

While mean field models frequently exhibit a finite number of multistable stationary solutions (see, e.g., the Desai–Zwanzig model [21] and the Takatsuji model [1, 33]), the linear discrepancy model discussed in section 2.1 and the Kuramoto model exhibit an infinitely large number of stationary solutions. The reason for this is the invariance of the respective many-body systems against translations. That is, if we replace the coordinate x in the linear discrepancy model (3) or in the Kuramoto model (22) by $x + \xi$ (where ξ is constant), then the evolution equations will maintain their structures and ξ eventually drops out of the equations. In this context, it might be worthwhile to consider further examples of translational invariant one-dimensional models for many-body systems with globally coupled subsystems. A famous example in this regard is the Fisher–Eigen model [45–48]. The state variable x of the Fisher–Eigen model describes the phenotype of species. Accordingly, $P(x, t)$ corresponds to the probability density of phenotypes observed at time t. Let us assume that the emergence and vanishing of phenotypes is determined by a selection process that results in a directed overall shift of phenotypes (see [47, section 8.2] or [49, 8.2.4]). Then the Fisher–Eigen model reads

$$\frac{\partial}{\partial t} P(x,t) = \alpha \left(x - \int_\Omega x' P(x',t)\,\mathrm{d}x' \right) P(x,t) + Q \frac{\partial^2}{\partial x^2} P(x,t). \qquad (48)$$

Note that the equation does not correspond to a drift-diffusion equation (or Fokker–Planck equation). Rather, the Fisher–Eigen model should be regarded as a reaction–diffusion model. Clearly, equation (48) exhibits translational invariance—just as equations (3) and (22). Equations (3), (22) and (48) are examples of a class of translational invariant many-body systems with globally coupled subsystems satisfying first-order dynamical evolution equations. For models of this class, the following general statement holds: let $p_0(x,t)$ be a particular solution, then $p(x,t) = p_0(x+\xi,t)$ for a constant ξ corresponds to a solution as well. We implicitly exploited this statement in section 2.2 while discussing stationary probability densities of the linear discrepancy model.

It should be pointed out that models for group behavior are not necessarily translational invariant. For example, the Takatsuji model does not exhibit the symmetry property of translational invariance [1, 33]. Let us briefly discuss implications of this observation. Both the Takatsuji model and the linear discrepancy model describe interacting agents and the emergence of multistability. However, the Takatsuji model predicts that groups finally settle down in either of two modes (e.g. an altruistic mode versus a selfish mode; a risky attitude versus a cautious attitude). That is, the Takatsuji model does not exhibit an infinitely large set of possible stationary distributions. In particular, the stationary distributions related to these two modes are completely defined by the model parameters. This discussion reveals that in general models for group behavior and group decision making may be classified into two types. Models that predict that statistical group properties depend on external circumstances conceptualized as model parameters and models that predict that the evolution of group properties is crucially influenced by the initial statistical group properties at the beginning of an experiment. Again, experiments may be designed and conducted in order to test under which conditions which of the two notions or two types of models reflect best group decision making.

References

[1] Takatsuji M 1975 *Biol. Cybern.* **17** 207
[2] Galam S and Zucker J D 2000 *Physica* A **287** 644
[3] Schweitzer F and Holyst J A 2000 *Eur. Phys. J.* B **15** 723
[4] Neda Z, Ravasz E, Brechet Y, Vicsek T and Barabasi A L 2000 *Nature* **403** 849
[5] Strogatz S H, Abrams D, McRobie A, Eckhardt B and Ott E 2005 *Nature* **438** 43
[6] Frank T D 2008 *J. Phys. A: Math. Theor.* **41** 282001
[7] Latane B and Nowak A 1994 *Behav. Sci.* **39** 1
[8] Latane B and Bourgeois M J 2001 *Social Influence: Direct and Indirect Processes* ed J P Forgas and K D Williams (Philadelphia: Psychology Press) pp 61–76
[9] Kalat J W 1993 *Introduction to Psychology* (Pacific Grove, CA: Brooks/Cole Publ. Company)
[10] Weiten W 1995 *Psychology: Themes and Variations* (Pacific Grove, CA: ITP)
[11] McKean H P Jr 1966 *PNAS* **56** 1907
[12] McKean H P Jr 1969 *Lectures in Differential Equations* vol II, ed A K Aziz (New York: Van Nostrand Reinhold Company) pp 177–93
[13] Dawson D A 1983 *J. Stat. Phys.* **31** 29
[14] Meleard S and Coppoletta S R 1987 *Stoch. Process. Appl.* **26** 317
[15] Meleard S 1996 *Probabilistic Models for Nonlinear Partial Differential Equations* ed C Graham, T G Kurtz, S Meleard, P E Potter, M Pulvirenti and D Talay (Berlin: Springer) pp 42–95
[16] Kolokoltsov V N 2007 *J. Stat. Phys.* **126** 585
[17] Stroock D W 2003 *Markov Processes from K. Ito's Perspective* (Princeton: Princeton University Press)
[18] Frank T D 2008 *Phys. Lett.* A **372** 4553
[19] Frank T D 2008 *Phys. Lett.* A **372** 6235
[20] Frank T D 2009 *Encyclopedia of Complexity and Systems Science* ed R A Meyers (Berlin: Springer) (in press, to be published in 2009)
[21] Desai R C and Zwanzig R 1978 *J. Stat. Phys.* **19** 1

[22] Shiino M 1987 *Phys. Rev.* A **36** 2393
[23] Plastino A R and Plastino A 1995 *Physica* A **222** 347
[24] Kaniadakis G 2001 *Physica* A **296** 405
[25] Kuramoto Y 1984 *Chemical Oscillations, Waves, and Turbulence* (Berlin: Springer)
[26] Acebron J A, Bonilla L L, Vicente C J P, Ritort F and Spigler R 2005 *Rev. Mod. Phys.* **77** 137
[27] Chavanis P H 2003 *Phys. Rev.* E **68** 036108
[28] Borland L 1998 *Phys. Rev.* E **57** 6634
[29] Curado E M F and Nobre F D 2003 *Phys. Rev.* E **67** 021107
[30] Frank T D and Daffertshofer A 1999 *Physica* A **272** 497
[31] Frank T D, Daffertshofer A, Peper C E, Beek P J and Haken H 2001 *Physica* D **150** 219
[32] Frank T D 2004 *Physica* A **331** 391
[33] Frank T D 2005 *Nonlinear Fokker–Planck Equations: Fundamentals and Applications* (Berlin: Springer)
[34] Boster F J, Hunter J E, Mayer M E and Hale J L 1980 *Communication Yearbook 4* ed D Nimmo (New Brunswick, NJ: Transaction Books) pp 165–76
[35] Boster F J, Fryrear J E, Mongeau P A and Hunter J E 1982 *Communication Yearbook 6* ed M Burgoon (Beverly Hills, CA: Sage) pp 395–418
[36] Boster F J, Hunter J E and Hale J L 1991 *Commun. Res.* **18** 524
[37] Risken H 1989 *The Fokker–Planck Equation—Methods of Solution and Applications* (Berlin: Springer)
[38] Tass P A 1999 *Phase Resetting in Medicine and Biology—Stochastic Modelling and Data Analysis* (Berlin: Springer)
[39] Zaks M A, Neiman A B, Feistel S and Schimansky-Geier L 2003 *Phys. Rev.* E **68** 066206
[40] Kawai R, Sailer X and Schimansky-Geier L 2004 *Phys. Rev.* E **69** 051104
[41] Shiino M and Yoshida K 2001 *Phys. Rev.* E **63** 026210
[42] Zaks M A, Schimansky-Geier L and Neiman A B 2005 *Chaos* **15** 026117
[43] Ichiki A, Ito H and Shiino M 2007 *Physica* E **40** 402
[44] Frank T D, Daffertshofer A and Beek P J 2001 *Phys. Rev.* E **63** 011905
[45] Fisher R A 1930 *The Genetical Theory of Natural Selection* (Oxford: Oxford University Press)
[46] Eigen M 1958 *Naturwissenschaften* **58** 465
[47] Feistel R and Ebeling W 1989 *Evolution of Complex System: Self-Organization, Entropy and Development* (Berlin: VEB Verlag)
[48] Dunkel J, Ebeling W, Schimansky-Geier L and Hänggi P 2003 *Phys. Rev.* E **67** 061118
[49] Timofeeff-Ressovsky N V, Voroncov N N and Jablokov A N 1975 *Kurzer Grundrißder Evolutionstheorie* (Jena: VEB Verlag) (in German)